# Convex approximations for multistage stochastic mixed-integer programs

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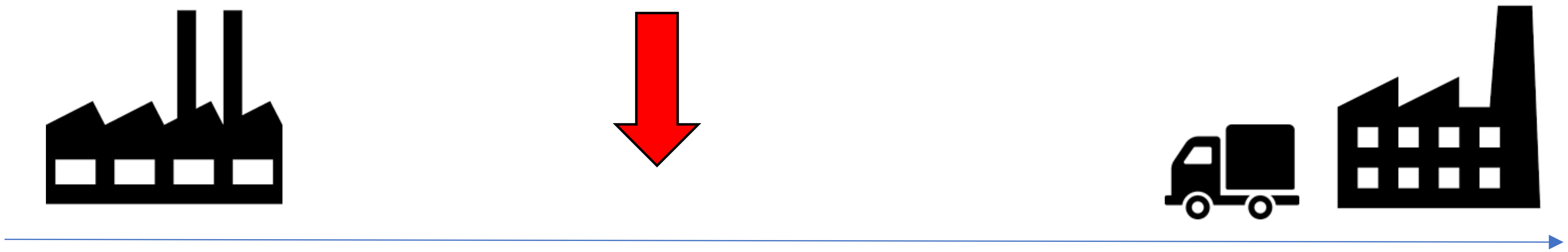
# Production planning example

- Timeline:

Observe demand $\omega$



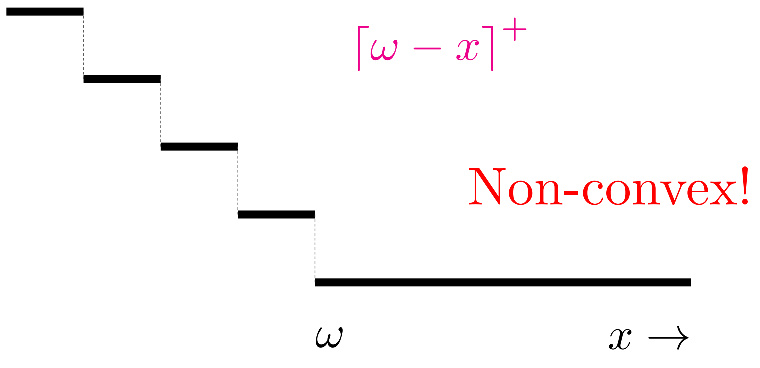
Produce $x \in \mathbb{R}_+$

If $\omega > x$, then buy $y \in \mathbb{Z}_+$ from competitor in batches of size 1

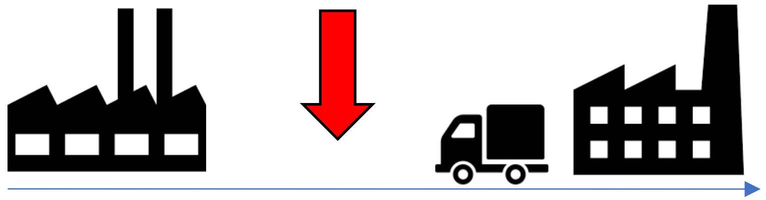
- Minimize total expected costs:

$$\min_{x \geq 0} \quad \underbrace{cx}_{\text{production costs}} + \underbrace{q\,\mathbb{E}_\omega\Big[\underbrace{\lceil \omega - x \rceil^+}_{y = \max\{\lceil \omega - x \rceil, 0\}}\Big]}_{\text{expected future purchasing costs}}$$

- For fixed demand $\omega$:



# Non-convex objective function?



- Minimize total expected costs:

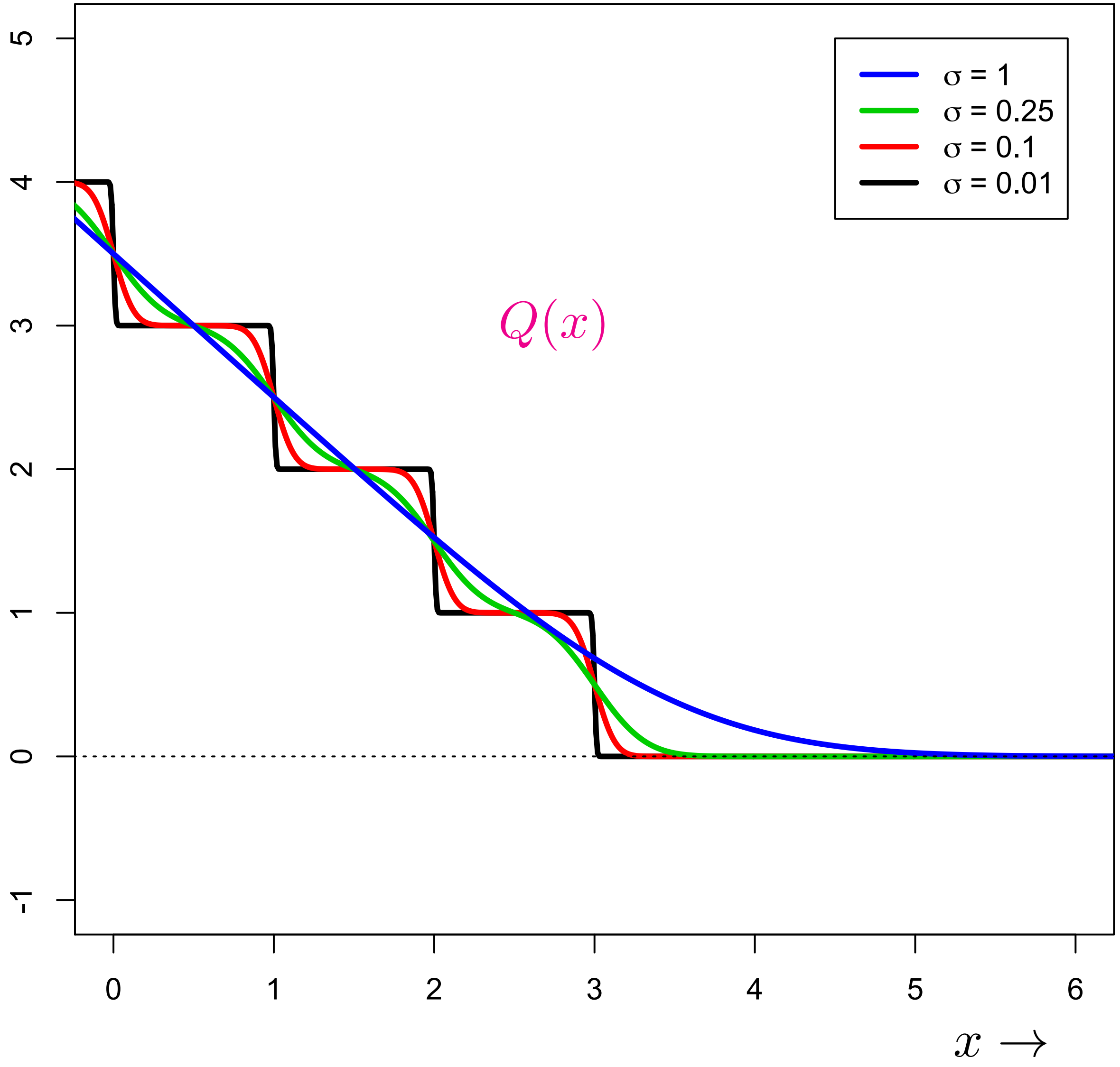
$$\min_{x\geq 0} \; cx + \underbrace{q\,\mathbb{E}_\omega\Big[\lceil \omega - x\rceil^+\Big]}_{Q(x)}$$

- Numerical example:

$$\omega \sim N(\mu, \sigma^2)$$

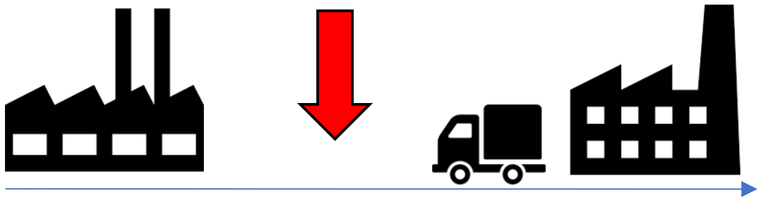
$$\mu = 3$$

$$q = 1$$




Observation: $Q$ “convexer”
if $\sigma$ increases

# Bounds on the expectation of periodic functions



- Convex approximation $\hat{Q}$

$$\hat{Q}(x) = \mathbb{E}_\omega\left[(\omega + 1/2 - x)^+\right]$$

- Approximation error

$$\hat{Q}(x) - Q(x) = \mathbb{E}_\omega[\varphi_x(\omega)]$$

- Error bound

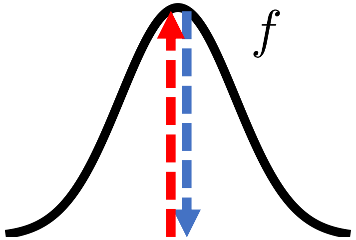
For any $\varphi : \mathbb{R} \to \mathbb{R}$ and random variable $\omega$ with pdf $f$

$$\left|\mathbb{E}_\omega[\varphi(\omega)]\right| \leq \frac{|\Delta| f}{2} \sup_{s \leq t} \left| \int_s^t \varphi(u) du \right|$$

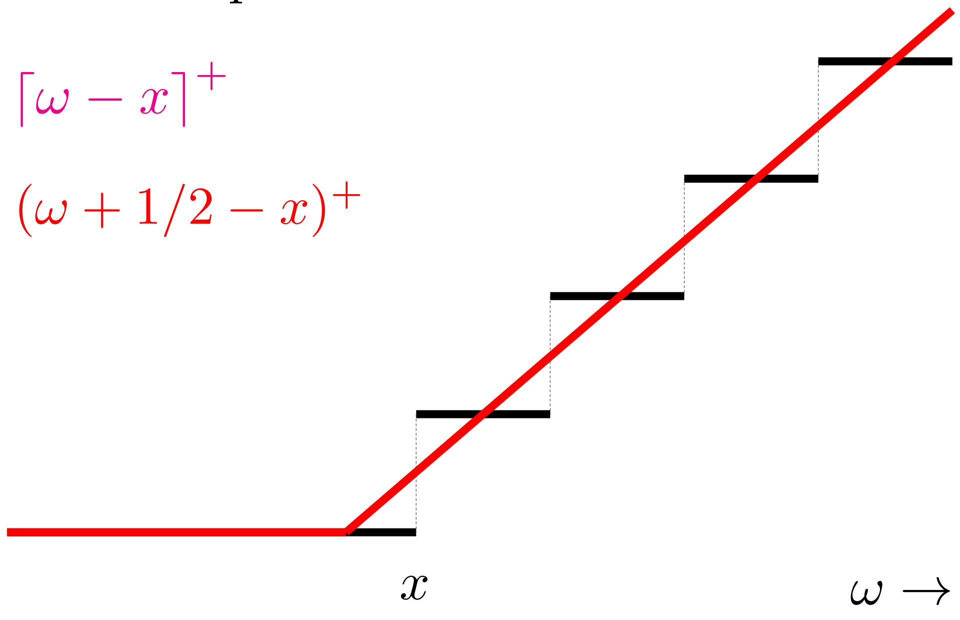
with $|\Delta| f$ denoting the total variation of $f$

- Total variation

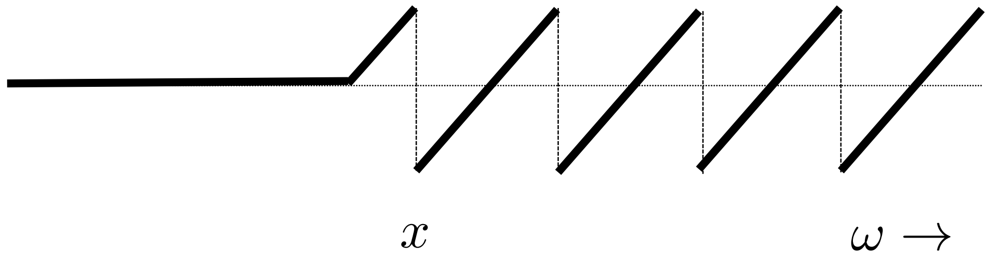
$$|\Delta| f = \text{total increase} + \text{total decrease}$$




- For a fixed production level $x$:




- Periodic difference function $\varphi_x(\omega)$



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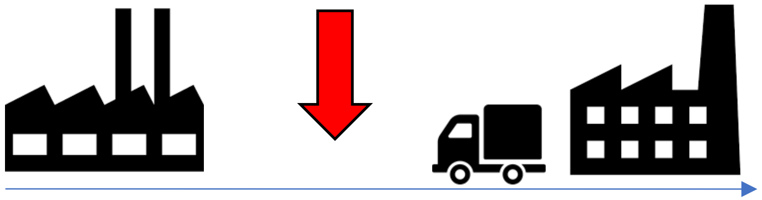

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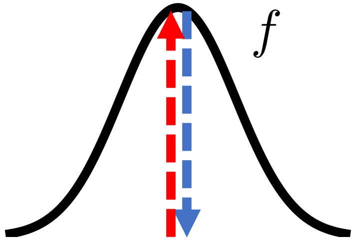


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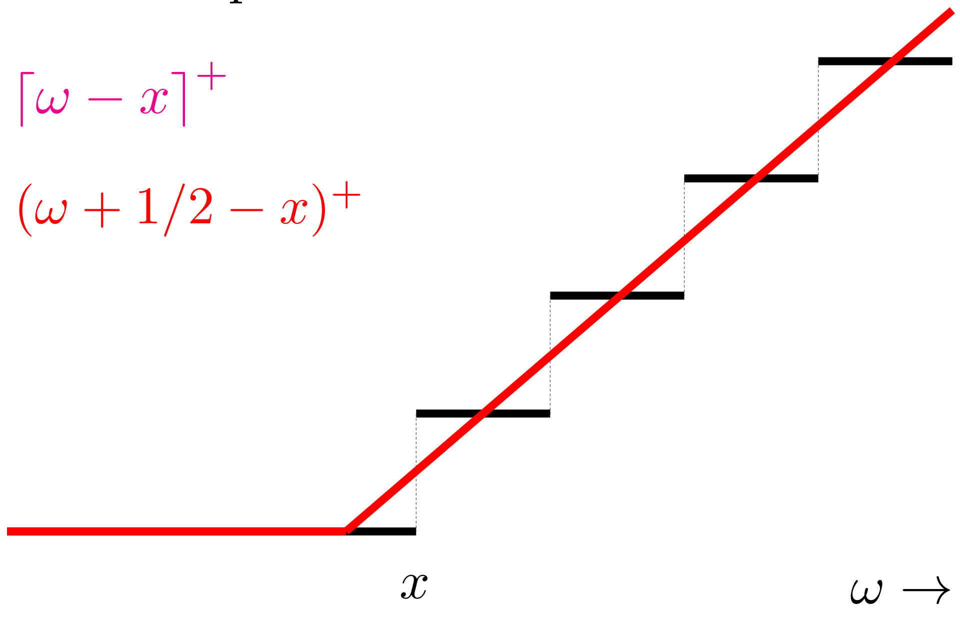


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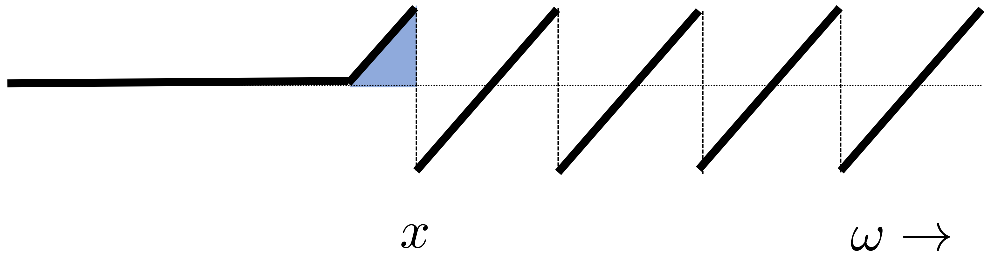

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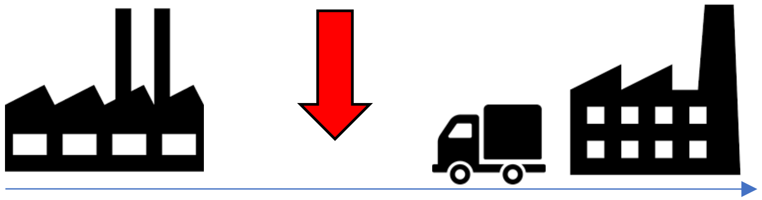

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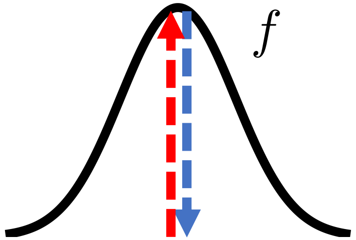



- Goal

- Production planning example is a stochastic MIP

- Prove asymptotic convexity for multistage stochastic MIPs

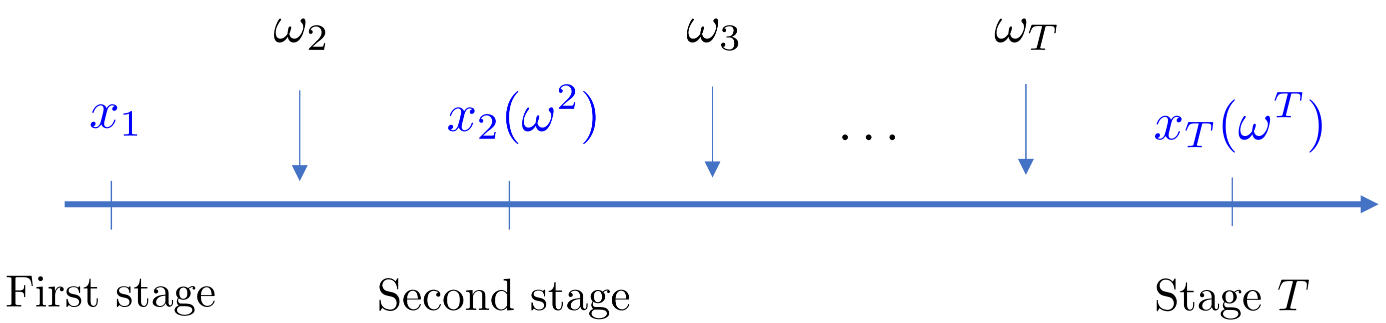
# Convex approximations for multistage stochastic mixed-integer programs

## Outline

- Definition multistage stochastic mixed-integer programs (M-SMIPs)
  - Nested formulation
  - Known results for two-stage SMIPs
- Asymptotic periodicity for mixed-integer value functions
- Construction convex approximations for M-SMIPs
- Asymptotic total variation error bound

# Multistage stochastic mixed-integer programs (M-SMIPs)

- Notation:
  - Time horizon $t = 1, \dots, T$
  - Decision variables $x_t(\omega^t)$
  - Random parameters $\omega_t$ with history $\omega^t := (\omega_1, \dots, \omega_t)$




- Nested formulation:

Fixed

$$\eta := \min_{x_1 \in X_1} \left\{ c_1^\top x_1 + \underbrace{Q_1(x_1)} : W_1 x_1 = \omega_1 \right\}$$

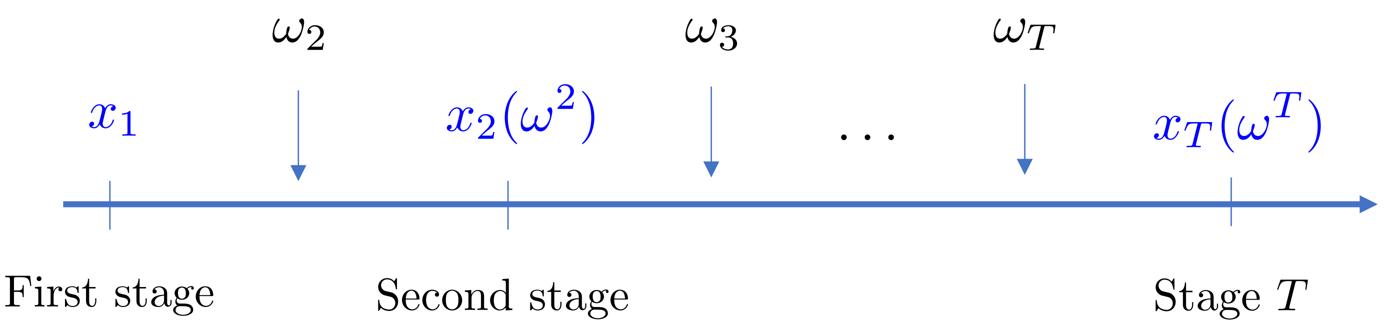
$$Q_t(x_t) = \mathbb{E}_{\omega_{t+1}} \Big[ \underbrace{v_{t+1}(\omega_{t+1}, x_t)} \Big], \qquad t = 1, \dots, T-1 \qquad \text{with } Q_T \equiv 0$$

Integrality restrictions

$$v_t(\omega_t, x_{t-1}) = \min_{x_t \in X_t} \left\{ c_t^\top x_t + Q_t(x_t) : T_t x_{t-1} + W_t x_t = \omega_t \right\}$$

# Multistage stochastic mixed-integer programs (M-SMIPs)

- Assumptions:
  - Sufficiently expensive and complete recourse
  
  $\longrightarrow$ $v_t$ is always finite
  - Stagewise independent $\omega_t$
  - $\mathbb{E}_{\omega_t}\|\omega_t\| < +\infty \longrightarrow Q_t$ is finite
  - The matrix of coefficients $W_t$ is integer

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Fixed

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$$Q_t(x_t) = \mathbb{E}_{\omega_{t+1}}\left[ v_{t+1}(\omega_{t+1}, x_t) \right], \qquad t = 1, \dots, T-1 \qquad \text{with } Q_T \equiv 0$$

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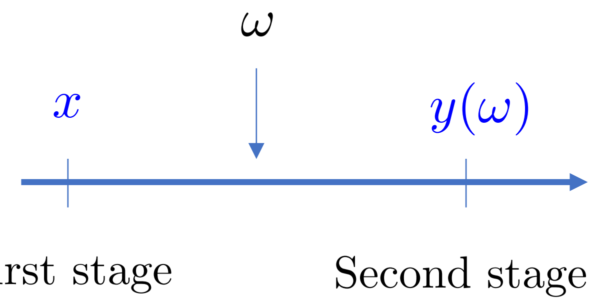
# Two-stage stochastic MIPs

- Stage-$t$ value function

$$v_t(\omega_t, x_{t-1}) = \min_{x_t \in X_t} \left\{ c_t^\top x_t + Q_t(x_t) : T_t x_{t-1} + W_t x_t = \omega_t \right\}$$

- Second-stage value function:

$$\underset{\substack{\uparrow \\ 1^{\text{st}}\text{-stage variables}}}{v(\omega, x)} = \min_y \left\{ q^\top y : \underset{\substack{\uparrow \\ 2^{\text{nd}}\text{-stage variables}}}{W y} = \omega - Tx,\ y \in \underbrace{\mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3}}_{2^{\text{nd}}\text{-stage feasible region}} \right\}$$

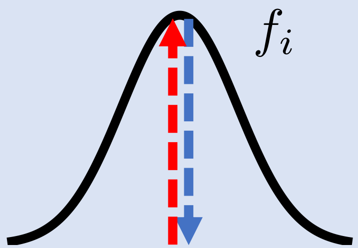



- First-stage expected value function:

$$Q(x) = \mathbb{E}_\omega[v(\omega, x)]$$

- Theorem (R., Schultz, van der Vlerk and Klein Haneveld, 2016)
  - There exists a convex approximation $\hat{Q}(x) = \mathbb{E}_\omega[\hat{v}(\omega, x)]$ of $Q(x)$ and a constant $C$ such that for all independent random vectors $\omega$ with marginal density functions $f_i$

$$\|Q - \hat{Q}\|_\infty \leq C \sum_{i=1}^{m} |\Delta| f_i$$



# Generic continuous value function

- Second-stage value function:

$$v(\omega, x) = \min_y \left\{ q^\top y : Wy = \omega - Tx,\ \ y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3} \right\}$$

- LP-relaxation:

$$v_{LP}(\omega, x) = \min_y \left\{ q^\top y : Wy = \omega - Tx,\ y \geq 0 \right\}$$

- For any basis matrix $B$:

$$v_{LP}(\omega, x) = \min_{y_B, y_N} \left\{ q_B^\top y_B + q_N^\top y_N \,:\, By_B + Ny_N = \omega - Tx,\ y_B \geq 0,\ y_N \geq 0 \right\}$$

- Substituting $y_B = B^{-1}(\omega - Tx) - B^{-1}Ny_N$:

$$\begin{aligned} v_{LP}(\omega, x) = q_B^\top B^{-1}(\omega - Tx) + \min_{y_N} \ \ & \bar{q}_N^\top y_N \\ \text{s.t.} \quad & B^{-1}(\omega - Tx) - B^{-1}Ny_N \geq 0 \\ & y_N \geq 0 \end{aligned}$$

- with reduced costs $\bar{q}_N^\top := q_B^\top - q_N^\top B^{-1} N \geq 0$

# Generic continuous value function

$$
\begin{aligned}
v_{LP}(\omega, x) = q_B^\top B^{-1}(\omega - Tx) + \min_{y_N} \quad & \bar{q}_N^\top y_N \\
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& y_N \geq 0
\end{aligned}
$$

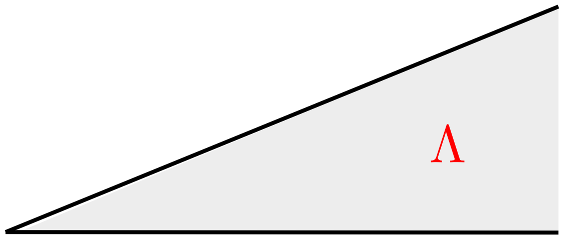
with reduced costs $\bar{q}_N^\top := q_B^\top - q_N^\top B^{-1} N \geq 0$

- Observation:

$$
y_N^*(\omega, x) = 0 \text{ if } \underbrace{B^{-1}(\omega - Tx) \geq 0}
$$

$$
\longrightarrow \omega - Tx \in \Lambda := \left\{ s \in \mathbb{R}^m : B^{-1} s \geq 0 \right\}
$$

closed convex cone



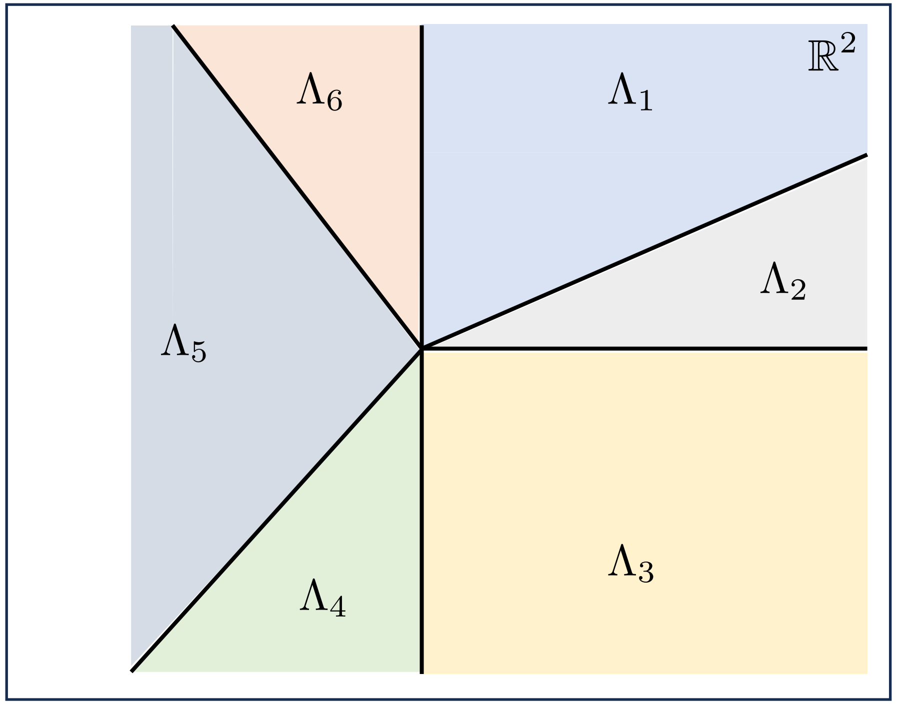

- Holds for all basis matrices $B$ with reduced costs $\bar{q}_N^\top \geq 0$

$\longrightarrow$ Basis Decomposition Theorem

# Basis decomposition theorem

$$v_{LP}(\omega, x) = q_B^\top B^{-1}(\omega - Tx) + \min_{y_N} \ \bar{q}_N^\top y_N$$
$$\text{s.t.} \quad B^{-1}(\omega - Tx) - B^{-1} N y_N \geq 0$$
$$y_N \geq 0$$

with reduced costs $\bar{q}_N^\top := q_B^\top - q_N^\top B^{-1} N \geq 0$




- Walkup and Wets (1969)
  - There exist
    - dual feasible basis matrices $B^k$
    - corresponding closed convex cones $\Lambda_k$ with $\bigcup_{k=1}^{K} \Lambda_k = \mathbb{R}^m$ and $\text{int}(\Lambda_i) \cap \text{int}(\Lambda_j) = \emptyset$
  - such that

$$v_{LP}(\omega, x) = \underbrace{q_{B^k}^\top (B^k)^{-1}(\omega - Tx)}_{\text{affine}} \qquad \text{if } \omega - Tx \in \Lambda_k$$

# Mixed-integer value function

$$v(\omega, x) = q_B^\top B^{-1}(\omega - Tx) + \min_{y_N} \ \bar{q}_N^\top y_N$$
$$\text{s.t.} \quad B^{-1}(\omega - Tx) - B^{-1} N y_N \in \mathbb{Z}_+^{n_B} \times \mathbb{R}_+^{m-n_B}$$
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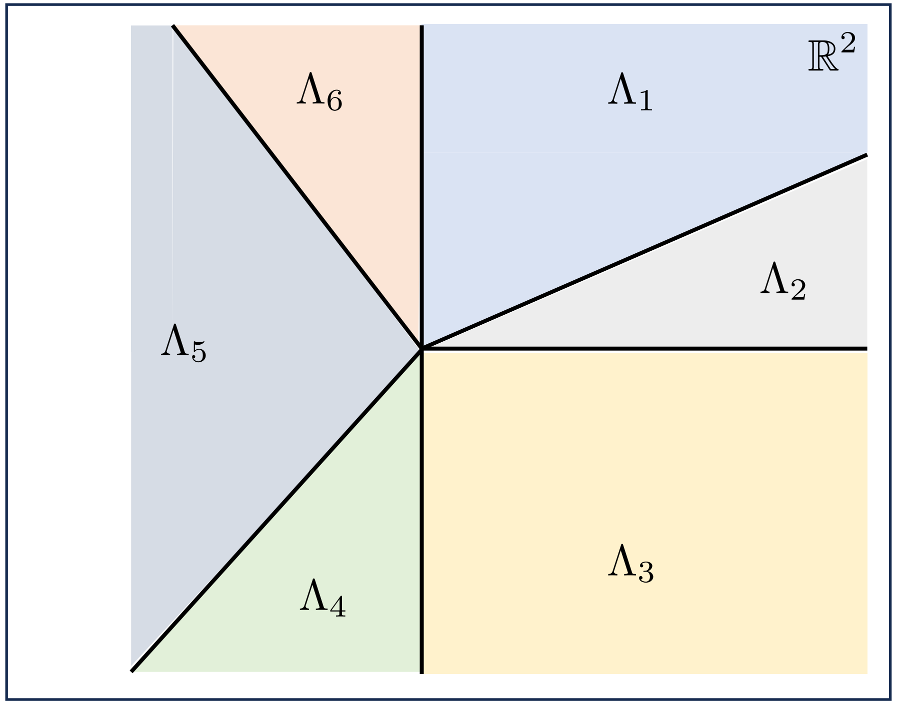


- Gomory (1969) and R. et al. (2016)
  - There exist
    - dual feasible basis matrices $B^k$
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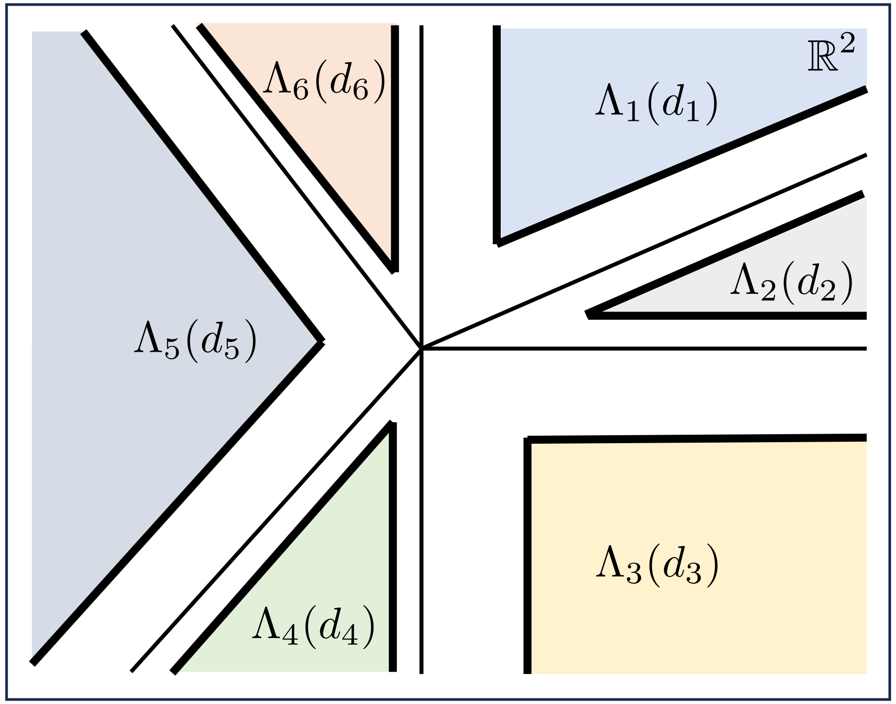
    - distances $d_k$
    - periodic functions $\psi_k$
  - such that

Points in $\Lambda_k$ with at least distance $d_k$ to the boundary of $\Lambda_k$

$$v(\omega, x) = \underbrace{q_{B^k}^\top (B^k)^{-1}(\omega - Tx)}_{\text{affine}} + \underbrace{\psi_k(\omega, x)}_{\text{periodic}} \qquad \text{if } \omega - Tx \in \Lambda_k(d_k)$$

# Mixed-integer value function

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\begin{aligned}
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with reduced costs $\bar{q}_N^\top := q_B^\top - q_N^\top B^{-1} N \geq 0$

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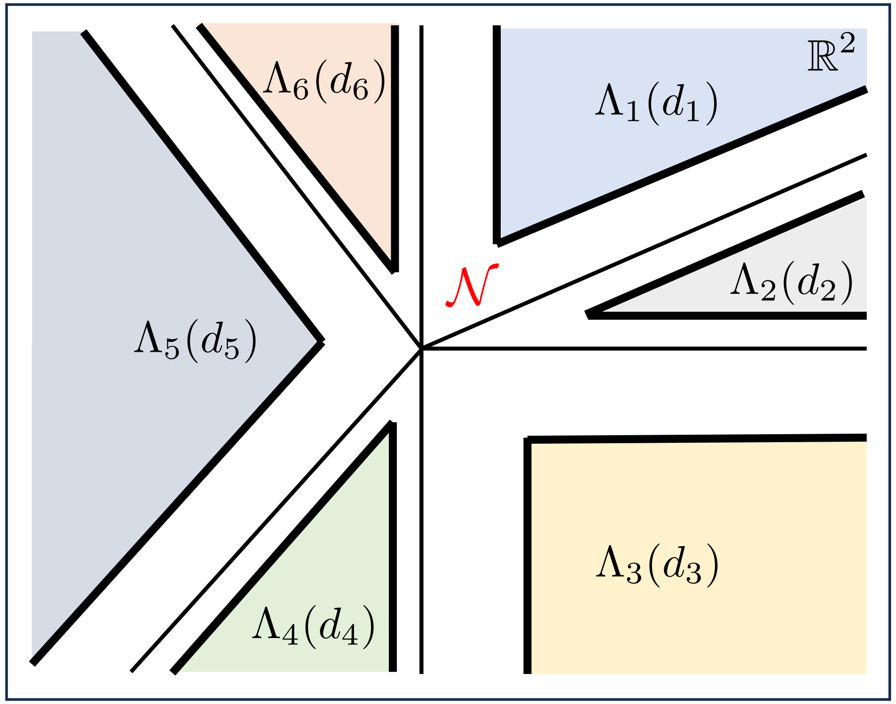
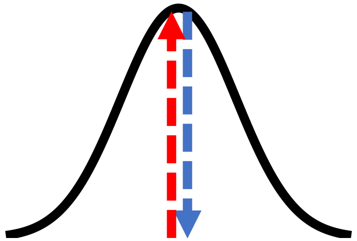
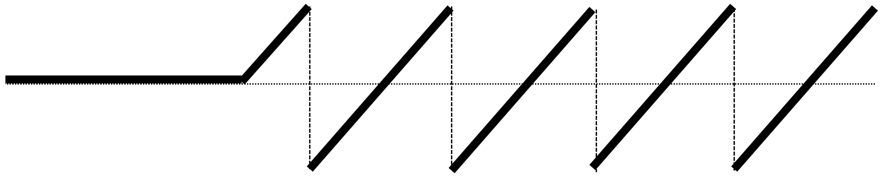



- Intuition asymptotic convexity $Q$
  - Area $\mathcal{N} := \bigcup_{k=1}^{K} \left( \Lambda_k \backslash \Lambda_k(d_k) \right)$ is "small"
  - For every $\Lambda_k(d_k)$:
    - Total variation error bound for periodic functions





# Multistage SMIPs

- Stage-$t$ value function

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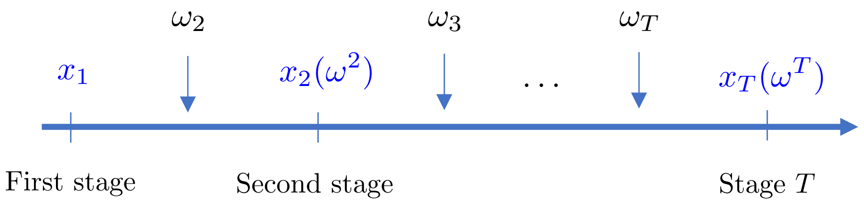



- Generic value function:

$$v(\omega, x) = \min_y \left\{ q^\top y + \underbrace{Q(y)}_{\text{Expected cost-to-go function}} : Wy = \omega - Tx,\ y \in \underbrace{\mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3}}_{\text{2nd-stage feasible region}} \right\}$$

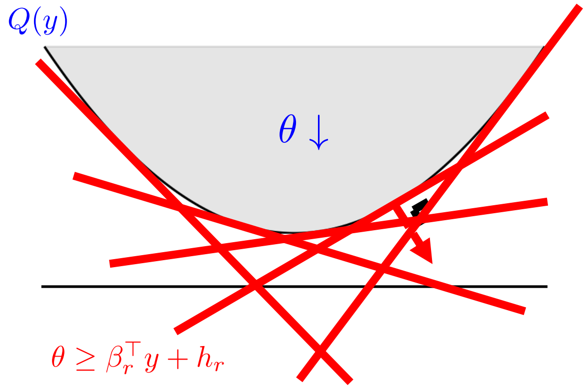
1st-stage variables

2nd-stage variables

- Assumption:
  - Expected cost to-go-function $Q$ is convex polyhedral
    - Holds if $x_\tau$ is continuous and $\omega_\tau$ discrete for $\tau = t+1, \dots, T$
    - Holds if $Q$ can be approximated well by a convex approximation $\hat{Q}$

$$\longrightarrow Q(y) = \min_\theta \ \theta \quad \text{s.t.} \ \theta \geq \beta_r^\top y + h_r \quad \forall r = 1, \dots, R$$

does not depend on $\omega$ and $x$




- Generic mixed-integer value function:

$$v(\omega, x) = \min_y \left\{ q^\top y : Wy = \begin{pmatrix} h \\ \omega - Tx \end{pmatrix}, y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3} \right\}$$

# Generic mixed-integer value function

- Definition:

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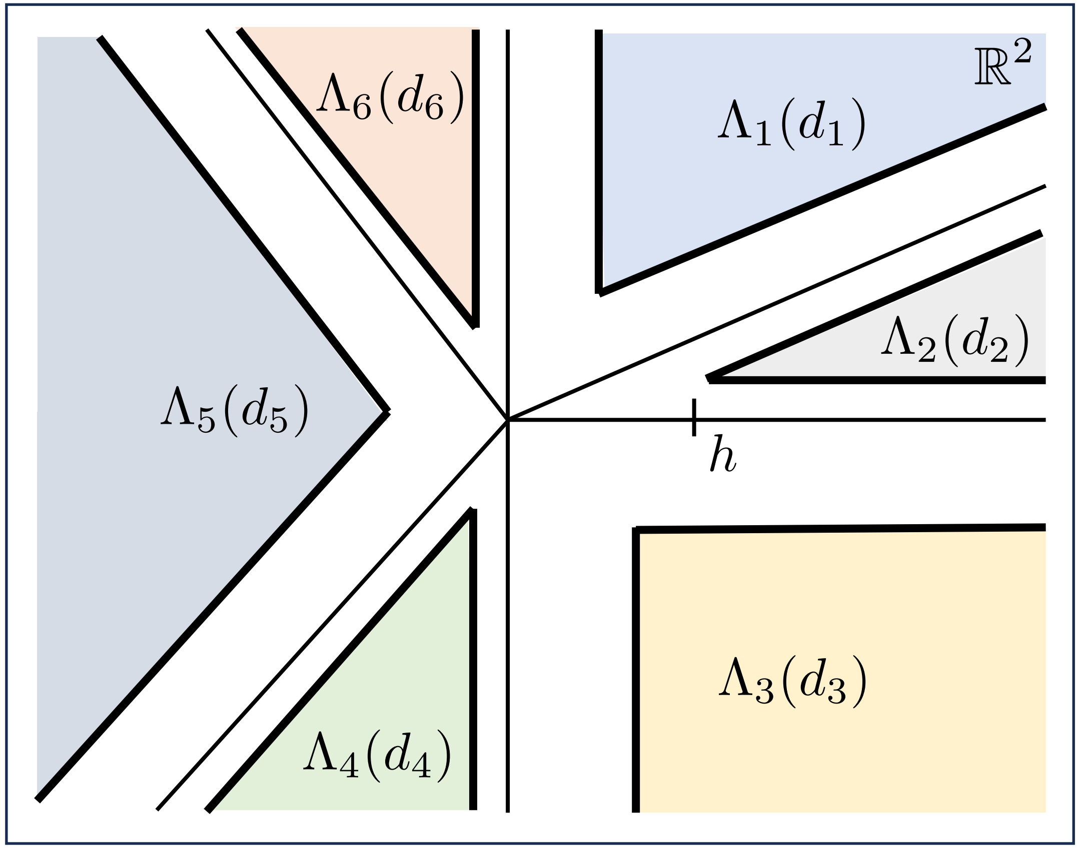

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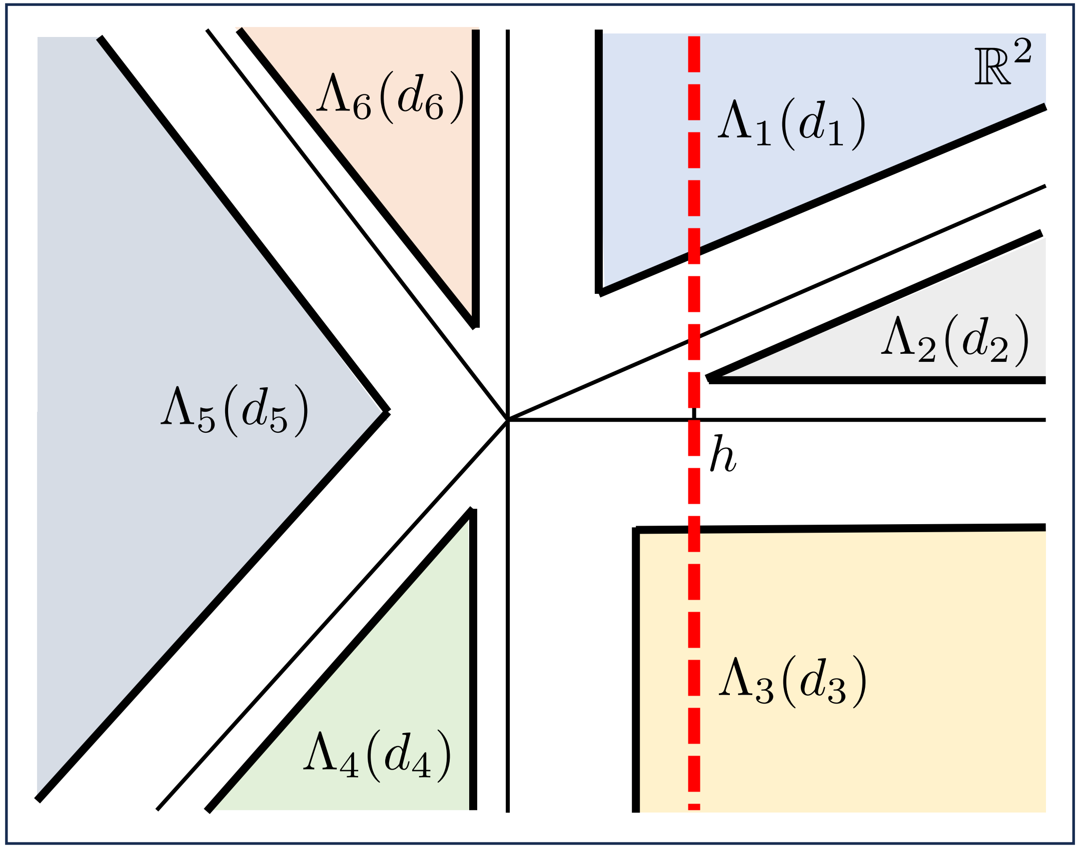


For fixed $x$ and $h$

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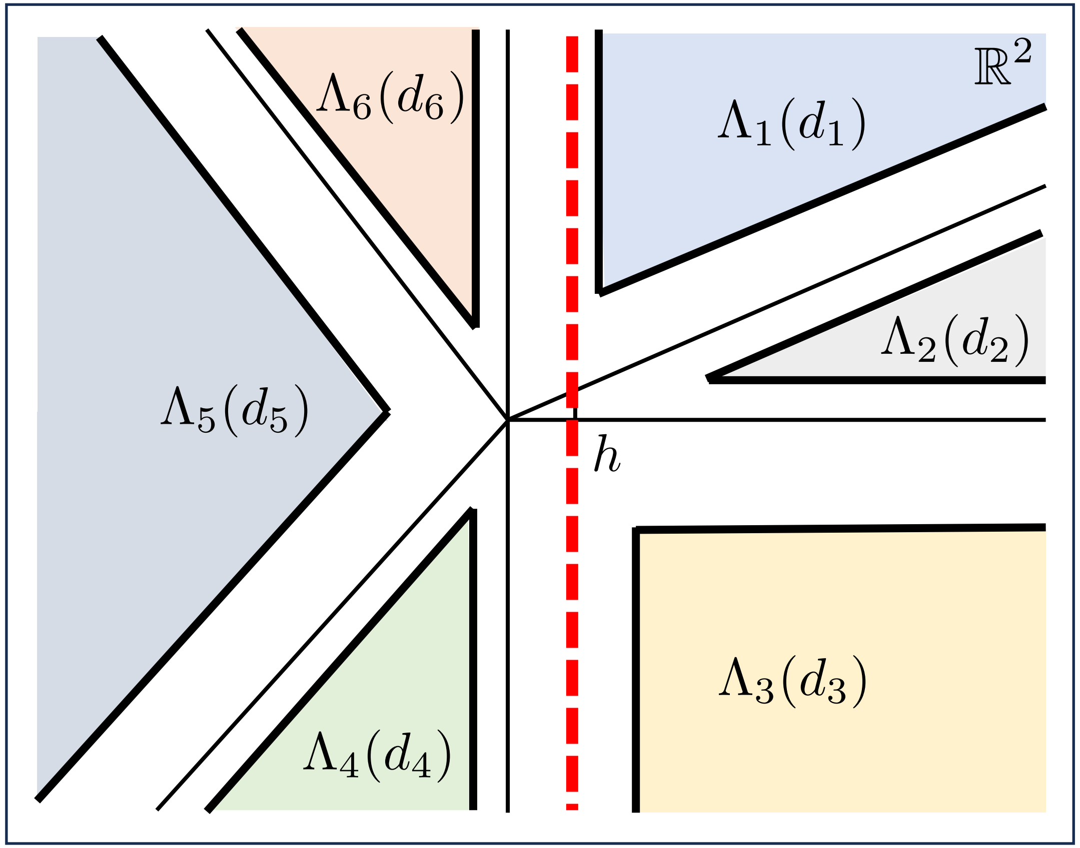
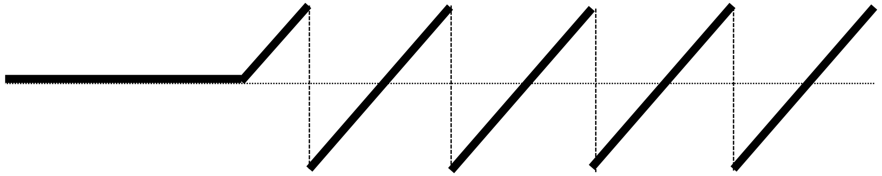



For fixed $x$ and $h$

- Observations:
  - Need to derive additional properties of $v(\omega, x)$ when right-hand side is partially uncertain

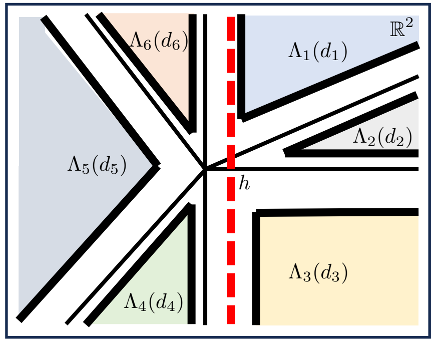
- Next steps:
  - Prove asymptotic periodicity for $v(\omega, x)$ with partial r.h.s. uncertainty
    - Requires new Adapted Gomory relaxation



# Gomory relaxation

$$
\begin{aligned}
v(\omega, x) = q_B^\top B^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \min_{y_N} \quad & \bar{q}_N^\top y_N \\
\text{s.t.} \quad & B^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} - B^{-1} N y_N \in \mathbb{Z}_+^{n_B} \times \mathbb{R}_+^{m-n_B} \\
& y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}
\end{aligned}
$$

- Identify critical basic variables $(y_B)_i$:
  - Write $B^{-1} = \begin{pmatrix} B_h^{-1} & B_\omega^{-1} \end{pmatrix} \quad \longrightarrow y_B = B_h^{-1} h + B_\omega^{-1}(\omega - Tx) - B^{-1} N y_N$
  - Define $i \in I \quad \Leftrightarrow \quad (B_\omega^{-1})_i = 0$




- Adapted Gomory relaxation $v_B(\omega, x)$:
  - Relax non-negativity of $(y_B)_i$ if $i \notin I$

$$
\begin{aligned}
v_B(\omega, x) = q_B^\top B^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \min_{y_N} \quad & \bar{q}_N^\top y_N \\
\text{s.t.} \quad & B^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} - B^{-1} N y_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B} \\
& e_i^\top B_h^{-1} h - e_i^\top B^{-1} N y_N \geq 0 \quad \forall i \in I \\
& y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}
\end{aligned}
$$

$i$-th unit vector

# Adapted Gomory relaxation

- Adapted Gomory relaxation:

$$
\begin{aligned}
v_B(\omega, x) = q_B^\top B^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \min_{y_N} \quad & \bar{q}_N^\top y_N \\
\text{s.t.} \quad & B^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} - B^{-1} N y_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B} \\
& e_i^\top B_h^{-1} h - e_i^\top B^{-1} N y_N \geq 0 \quad \forall i \in I \\
& y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}
\end{aligned}
$$

- Properties of optimal solutions $y_N^\star(\omega, x)$:
  - For every $x$:
    - $|\det B|$-periodic in $\omega$
    - uniformly bounded
    - Optimal for $v(\omega, x)$ if $(y_B^\star(\omega, x))_i \geq 0$ for all $i \notin I$

# Periodicity of $y_N^*(\omega, x)$

- Adapted Gomory relaxation:

$$\begin{aligned} v_B(\omega,x) = q_B^\top B^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix} + \min_{y_N} \quad & \bar{q}_N^\top y_N \\ \text{s.t.} \quad & B^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix} - B^{-1}Ny_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B} \\ & e_i^\top B_h^{-1}h - e_i^\top B^{-1}Ny_N \geq 0 \quad \forall i \in I \\ & y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N} \end{aligned}$$

- Definition
  - A function $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is called $p$-periodic for some $p \in \mathbb{R}$ if and only if

    $\varphi(s + pl) = \varphi(s)$ for all $s \in \mathbb{R}^m$ and $l \in \mathbb{Z}^m$

- $|\det B|$-periodicity of $y_N^\star(\omega, x)$ in $\omega$:
  - Let $\omega' = \omega + |\det B| l$ with $l \in \mathbb{Z}^m$

$$\longrightarrow \quad B^{-1}\begin{pmatrix} h \\ \omega' - Tx\end{pmatrix} = B^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix} + \underbrace{B^{-1}\begin{pmatrix} 0 \\ |\det B| l\end{pmatrix}}_{\text{Since } W \text{ is integer} \;\rightarrow\; \frac{1}{\det B}\operatorname{adj}(B)\begin{pmatrix} 0 \\ |\det B| l\end{pmatrix} \in \mathbb{Z}^m}$$

  - Fractional values of $B^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix}$ and $B^{-1}\begin{pmatrix} h \\ \omega' - Tx\end{pmatrix}$ are the same
  - Thus, $y_N^\star(\omega, x) = y_N^\star(\omega + pl, x)$ for $p = |\det B|$ and for all $l \in \mathbb{Z}^m$

# Boundedness of $y_N^\star(\omega, x)$

- Adapted Gomory relaxation:

$$\begin{aligned} v_B(\omega,x) = q_B^\top B^{-1}\begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \min_{y_N} \quad & \bar{q}_N^\top y_N \\ \text{s.t.} \quad & B^{-1}\begin{pmatrix} h \\ \omega - Tx \end{pmatrix} - B^{-1} N y_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B} \\ & e_i^\top B_h^{-1} h - e_i^\top B^{-1} N y_N \geq 0 \quad \forall i \in I \\ & y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N} \end{aligned}$$

- Observations:
  - $\bar{q}_N^\top \geq 0$
  - For the LP-relaxation: $y_N^\star(\omega, x) = 0$ if $e_i^\top B_h^{-1} h \geq 0 \quad \forall i \in I$
    $\longrightarrow y_N^\star(\omega, x) = 0$ is feasible

- Theorem (e.g., Schrijver Thm 17.2 adapted to MIP):
  - Optimal solutions of MIPs and LP-relaxations are "close"
  - There exists a constant $D > 0$ such that for all $\omega$ and $x$: $\|y_N^\star(\omega, x)\| \leq D$

# Relation between $v(\omega, x)$ and $v_B(\omega, x)$

- Adapted Gomory relaxation:

$$
\begin{aligned}
v_B(\omega,x) = q_B^\top B^{-1}\begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \min_{y_N} \quad & \bar{q}_N^\top y_N \\
\text{s.t.} \quad & B^{-1}\begin{pmatrix} h \\ \omega - Tx \end{pmatrix} - B^{-1}N y_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B} \\
& e_i^\top B_h^{-1} h - e_i^\top B^{-1} N y_N \geq 0 \quad \forall i \in I \\
& y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}
\end{aligned}
$$

- Optimality condition:

  - If $\underbrace{e_i^\top B_h^{-1} h + e_i^\top B_\omega^{-1}(\omega - Tx) - e_i^\top B^{-1} N y_N^\star(\omega, x) \geq 0}_{(y_B^\star(\omega,x))_i \geq 0}$ for all $i \notin I$

  then $y_N^\star(\omega, x)$ is optimal for $v(\omega, x)$

- Sufficient condition:

  - Use boundedness of $y_N^\star(\omega, x)$

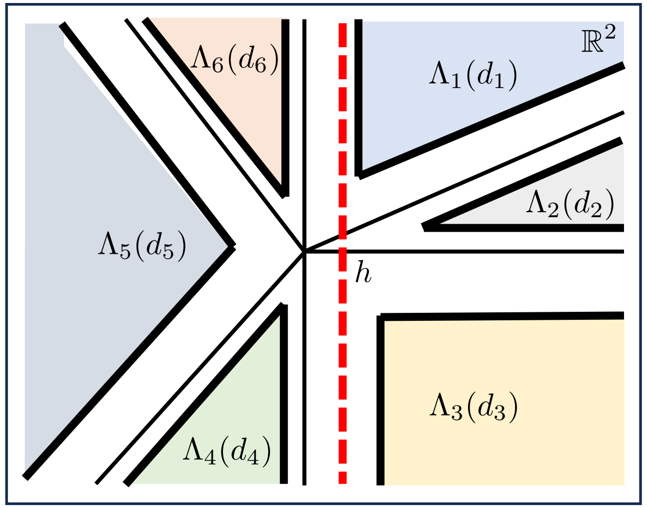
$$\hat{h}_i := \max_{\|y_N\| \leq D} e_i^\top B^{-1} N y_N - e_i^\top B_h^{-1} h$$

  - If $\omega - Tx \in \mathcal{H}_i(\hat{h}_i)$ for all $i \notin I$ with $\mathcal{H}_i(\hat{h}_i) := \left\{ s \in \mathbb{R}^m : e_i^\top B_\omega^{-1} s \geq \hat{h}_i \right\}$

# Relation between $v(\omega, x)$ and $v_B(\omega, x)$

- Adapted Gomory relaxation:

$$v_B(\omega,x) = q_B^\top B^{-1}\begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \min_{y_N} \ \bar{q}_N^\top y_N$$
$$\text{s.t.} \quad B^{-1}\begin{pmatrix} h \\ \omega - Tx \end{pmatrix} - B^{-1}Ny_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B}$$
$$e_i^\top B_h^{-1} h - e_i^\top B^{-1} N y_N \geq 0 \quad \forall i \in I$$
$$y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}$$




- Optimality condition:

  - If $\underbrace{e_i^\top B_h^{-1} h + e_i^\top B_\omega^{-1}(\omega - Tx) - e_i^\top B^{-1} N y_N^\star(\omega,x) \geq 0}_{(y_B^\star(\omega,x))_i \geq 0}$ for all $i \notin I$

  then $y_N^\star(\omega, x)$ is optimal for $v(\omega, x)$

- Sufficient condition:

  - Use boundedness of $y_N^\star(\omega, x)$

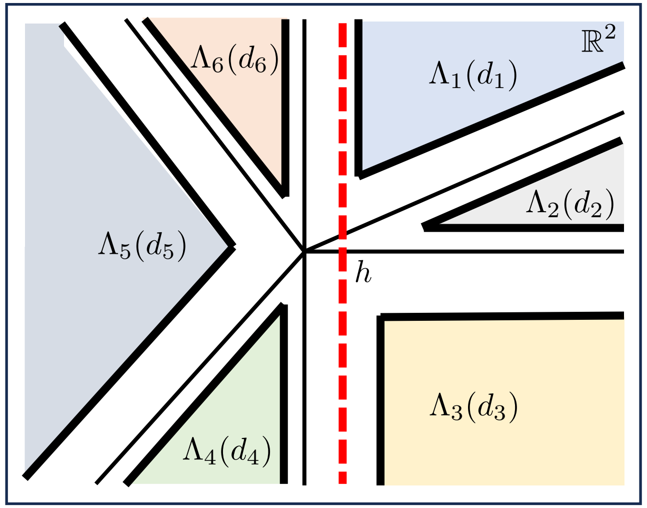
$$\hat{h}_i := \max_{\|y_N\| \leq D} e_i^\top B^{-1} N y_N - e_i^\top B_h^{-1} h$$

  - If $\omega - Tx \in \mathcal{H}_i(\hat{h}_i)$ for all $i \notin I$ with $\mathcal{H}_i(\hat{h}_i) := \left\{ s \in \mathbb{R}^m : e_i^\top B_\omega^{-1} s \geq \hat{h}_i \right\}$

# Relation between $v(\omega, x)$ and $v_B(\omega, x)$

- Adapted Gomory relaxation:

$$v_B(\omega,x) = q_B^\top B^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix} + \min_{y_N} \ \bar{q}_N^\top y_N$$
$$\text{s.t.} \quad B^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix} - B^{-1}N y_N \in \mathbb{Z}^{n_B} \times \mathbb{R}^{m-n_B}$$
$$e_i^\top B_h^{-1} h - e_i^\top B^{-1} N y_N \geq 0 \quad \forall i \in I$$
$$y_N \in \mathbb{Z}_+^{n_N} \times \mathbb{R}_+^{n-n_N}$$




- There exist $\tilde{\Lambda}_k(\tilde{d}_k) \subset \mathbb{R}^m$ such that

$$v(\omega,x) = \underbrace{q_{B_k}^\top B_k^{-1}\begin{pmatrix} h \\ \omega - Tx\end{pmatrix}}_{\text{affine}} + \underbrace{\bar{q}_{N_k}^\top y_{N_k}^\star(\omega,x)}_{\text{periodic}} \quad \text{if } \omega - Tx \in \tilde{\Lambda}_k(\tilde{d}_k)$$

- Define $\psi_k(\omega,x) := \bar{q}_{N_k}^\top y_{N_k}^\star(\omega,x)$

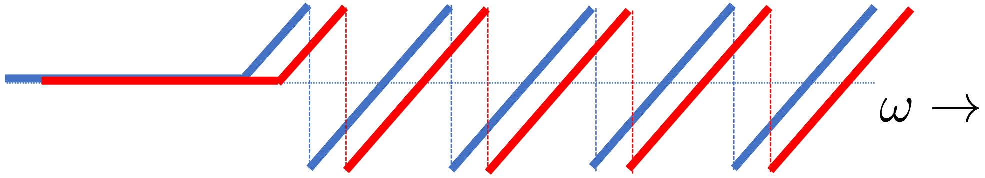
# Convex approximation

- Idea of convex approximation

$$v(\omega, x) = q_{B_k}^\top B_k^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \psi_k(\omega, x) \qquad \text{if } \omega - Tx \in \tilde{\Lambda}_k(\tilde{d}_k)$$

- Replace $x$ in $\psi_k(\omega, x)$ by a constant $\alpha$

$\longrightarrow$ $\psi_k(\omega, x) - \psi_k(\omega, \alpha)$ is periodic



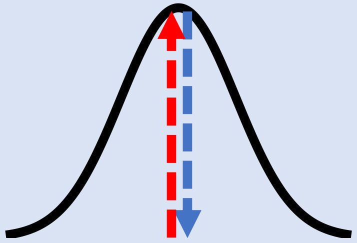

- Definition convex approximation $\hat{v}$

$$\hat{v}(\omega, x) = \max_{k=1,\dots,K} \left\{ q_{B^k}^\top (B^k)^{-1} \begin{pmatrix} h \\ \omega - Tx \end{pmatrix} + \psi_k(\omega, \alpha) \right\}$$

- Error bound (Independent case)

- There exists a constant $C > 0$ such that
  for all independent random vectors $\omega$ with marginal density functions $f_i$

$$\|Q - \hat{Q}\|_\infty \leq C \sum_{i=1}^{m} |\Delta| f_i$$



# Conclusion

- We derive a convex approximation and error bound for two-stage SMIP with partially uncertain right-hand side
- Can be applied to expected cost-to-go functions of M-SMIPs
  - By using induction over time stages
- Future research direction:
  - Construct SDDP-like algorithms based on convex approximation